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The dynamic contact problem for a prestressed cylindrical tube filled with a fluid☆

T.I. Belyankova, V.V. Kalinchuk

Rostov-on-Don, Russia

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ABSTRACT

The radial harmonic oscillations of a rigid bandage on the thin-walled elastic cylindrical tube filled with an ideal compressible fluid under a high static pressure are investigated. The problem is reduced to an integral equation, the kernel symbol of which is constructed in numerical form. The properties of the integral equation are investigated, a method of solving it is proposed, and the effect of the presence of the fluid and the initial stresses of the pipeline on the stress state in the contact area for dynamic actions are investigated. It is shown that when monitoring the initial stresses at high frequencies it is essential to take into account the presence of the fluid.

The mixed problem of the oscillations of a non-uniform cylinder, filled with an ideal fluid was considered previously in Refs 1 and 2, and its dynamic properties and the structure of the wave field on the free surface were investigated. The dynamic contact problem for a uniform cylinder filled with an ideal fluid was considered in the case when there were no initial stresses, and the distribution of the contact stresses and the structure of the surface wave field was investigated in Ref.3. Axisymmetric oscillations of a hollow composite cylinder, filled with Murnaghan material and subjected to the action of axial initial stresses were investigated, and the effect of the initial stresses and also the effect of taking into account third-order constants on the characteristics of longitudinal waves in a cylinder were considered in Ref.4.

1. Formulation of the problem

The problem of the radial oscillations of a bandage on the tube —a uniform infinite cylinder, filled with an ideal compressible fluid under a high static pressure, is considered. The inner and outer radii of the cylinder are equal to R_1 and R_0 respectively, ($(R_0-R_1)/R_0 \ll 1$), and the width of the bandage is 2a. The material of the cylinder walls is assumed to be elastic, initially isotropic.

We will consider the auxiliary problem in which the action of the bandage is replaced by the action of a harmonic load $\mathbf{q}(z)e^{-i\omega t}$, distributed in the region $|z| \le a$ on the cylinder surface.

We will assume that the oscillations are steady, i.e., all the quantities used to describe the problem can be represented in the form

$$f(r,\varphi,z,t) = f(r,\varphi,z)e^{-i\omega t} \tag{1.1}$$

The time factor will henceforth be omitted.

In cylindrical coordinates the boundary-value problem, taking representation (1.1) into account, has the form[1–6]

$$\frac{\partial\theta_{rr}}{\partial r} + \frac{\theta_{rr}-\theta_{\varphi\varphi}}{r} + \frac{1}{r}\frac{\partial\theta_{r\varphi}}{\partial\varphi} + \frac{\partial\theta_{rz}}{\partial z} = -\rho\omega^2 u_r$$

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E-mail address: kalin@rsu.ru (V.V. Kalinchuk).

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$$\frac{\partial\theta_{\varphi r}}{\partial r}+\frac{\theta_{r\varphi}+\theta_{\varphi r}}{r}+\frac{1}{r}\frac{\partial\theta_{\varphi\varphi}}{\partial\varphi}+\frac{\partial\theta_{\varphi z}}{\partial z} = -\rho\omega^2 u_\varphi, \quad \frac{\partial\theta_{zr}}{\partial r}+\frac{\theta_{zr}}{r}+\frac{1}{r}\frac{\partial\theta_{z\varphi}}{\partial\varphi}+\frac{\partial\theta_{zz}}{\partial z} = -\rho\omega^2 u_z$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\psi = -\frac{\omega^2}{C^2}\psi \tag{1.2}$$

$$r = R_0: \quad \mathbf{n}_r\cdot\boldsymbol{\Theta} = \begin{cases}\mathbf{q}(z), & |z|\le a\\ 0, & |z|>a\end{cases}$$

$$r = R_1: \quad \theta_{rr} = -i\rho_1\omega\psi, \quad -i\omega u_r = -\frac{\partial\psi}{\partial r}, \quad \theta_{rz} = \theta_{r\varphi} = 0, \quad |z|\le\infty \tag{1.3}$$

Here Θ is a linearized tensor, defining the stress state of the medium, θ_{rr}, θ_{rz}, $\theta_{r\varphi}$ are its components, $\mathbf{u} = \{u_r, u_\varphi, u_z\}$ is the displacement vector in the elastic medium, ρ is the density of the cylinder material, $\mathbf{n}_r$ is the normal to its side surface, C is the velocity of sound in the fluid, ρ_1 is its density, ψ is the potential and ω is the oscillation frequency.

The pressure of the fluid produces an axisymmetric initial stress state in the cylinder wall, non-uniform over the thickness. This leads to the need to use a system of linearized equations with variable coefficients, to enable us to take this fact into account.

2. The linearized equations of the oscillations of a prestressed non-uniform medium

When investigating the dynamics of a non-uniform continuous prestressed cylinder, the representation of the tensor $\boldsymbol{\Theta}$ in the form[7]

$$\boldsymbol{\Theta} = \mathbf{P}+\mathbf{U} \tag{2.1}$$

was used previously in Refs. [7,8]. Here $\mathbf{P}$ is a symmetrical tensor, determined by the state of the material and independent of the initial stresses. In the case of small initial strains it can be represented in the form (λ and μ are the Lamé parameters and $\mathbf{I}$ is the unit tensor)

$$\mathbf{P} = \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathbf{I}+2\mu\boldsymbol{\varepsilon}(\mathbf{u}) \tag{2.2}$$

and $\mathbf{U}$ is an antisymmetric tensor, related solely to the initial state[7]

$$\mathbf{U} = [\mathbf{T}\cdot\boldsymbol{\varepsilon}(\mathbf{u})-\boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathbf{T}]/2-\mathbf{T}\cdot\boldsymbol{\Omega}(\mathbf{u}) \tag{2.3}$$

$\boldsymbol{\varepsilon}(\mathbf{u})$ and $\boldsymbol{\Omega}(\mathbf{u})$ are symmetric and skew-symmetric strain tensors, and $\mathbf{T} = \mathbf{T}(r)$ is the initial stress tensor

$$\mathbf{T} = \begin{Vmatrix}\sigma_r^0 & 0 & 0\\ 0 & \sigma_\varphi^0 & 0\\ 0 & 0 & \sigma_z^0\end{Vmatrix} \tag{2.4}$$

Expression (2.1), taking formulae (2.2)–(2.4) into account in the axial-symmetry conditions, takes the form

$$\boldsymbol{\Theta} = \begin{Vmatrix}\theta_{rr} & \theta_{r\varphi} & \theta_{rz}\\ \theta_{\varphi r} & \theta_{\varphi\varphi} & \theta_{\varphi z}\\ \theta_{zr} & \theta_{z\varphi} & \theta_{zz}\end{Vmatrix} \tag{2.5}$$

$$\theta_{rr} = \lambda_m u_{r,r}+\lambda(r^{-1}u_r+u_{z,z}), \quad \theta_{zz} = \lambda_m u_{z,z}+\lambda(r^{-1}u_r+u_{r,r})$$

$$\theta_{\varphi\varphi} = \lambda_m r^{-1}u_r+\lambda(u_{r,r}+u_{z,z}), \quad \theta_{rz} = \mu^- u_{r,z}+\mu^+ u_{z,r}, \quad \theta_{zr} = \mu^- u_{r,z}+\mu^- u_{z,r}$$

$$\lambda_m = \lambda+2\mu, \quad \mu^\pm = \mu+(1/4\pm1/2)\sigma_r^0$$

After substituting expressions (2.5) into Eqs (1.2) and applying a Fourier transformation with respect to the z coordinate to the system obtained (α is the transformation parameter, and U_r, U_z and Ψ are the Fourier transforms of the functions u_r, u_z and ψ, respectively) we obtain the following system of ordinary differential equations

$$\lambda_m U_r''+l^+U_r'-(l^+r^{-1}+\mu^-\sigma_2^2)U_r-i\alpha\lambda^+U_z' = 0$$

$$-i\alpha\lambda^-U_r'-i\alpha(m^-+\lambda r^{-1})U_r+\mu^-U_z''+m^-U_z'-\lambda_m\sigma_1^2U_z = 0$$

$$\Psi''+r^{-1}\Psi'-\gamma^2\Psi = 0 \tag{2.6}$$

Here

$$l^+ = \frac{\lambda_m}{r}, \quad \sigma_1^2 = \alpha^2 - \frac{\rho\omega^2}{\lambda_m}, \quad \sigma_2^2 = \alpha^2 - \frac{\rho\omega^2}{\mu^-}$$

$$\lambda^\pm = \lambda + \mu^\pm, \quad m^- = (\mu^-)' + \frac{\mu^-}{r}, \quad \gamma^2 = \alpha^2 - \frac{\rho_1\omega^2}{C}$$

The boundary conditions take the form

$$r = R_0{:}\ \lambda_m U_r' + \lambda(r^{-1}U_r - i\alpha U_z) = Q_r(\alpha), \quad -i\alpha\mu^- U_r + \mu^+ U_z' = Q_z(\alpha)$$

$$r = R_1{:}\ \lambda_m U_r' + \lambda(r^{-1}U_r - i\alpha U_z) = i\rho_1\omega\Psi, \ -i\omega U_r = -\Psi', \ -i\alpha\mu^- U_r + \mu^+ U_z' = 0,$$

$$|z| \le \infty$$

$$\mathbf{Q}(\alpha) = \{Q_r, Q_z\} = \int_{-a}^{a} \mathbf{q}(z) e^{i\alpha z} dz \tag{2.7}$$

Eq. (2.6) represent a system of ordinary differential equations with variable coefficients, the solution of which, in analytic form, can only be obtained in exceptional cases, for example, when the moduli of elasticity and the density of the material of the medium both vary either trigonometrically or exponentially. In this paper the initial homogeneous stress state is described by a more complex function which makes it necessary to use numerical methods[9–17] to solve system (2.6) and to satisfy boundary conditions (2.7).

We will use dimensionless quantities below: linear quantities are referred to the tube wall thickness $h = R_0 - R_1$, and the stresses and forces in the elastic medium will be referred to the shear modulus μ. We will use the parameter $\kappa_2 = \omega h C_S^{-1}$ as the dimensionless frequency, the parameter $\rho' = \rho_1\rho^{-1}$ as the dimensionless fluid density, and the parameter $\nu = CC_S^{-1}$ as the dimensionless velocity of sound, where $C_S = \mu\rho^{-1}$ is the velocity of a shear wave in the cylinder wall.

3. The initial stress state of the cylinder filled with a fluid under pressure

To determine the initial stress state we will start from the fact that the cylinder is infinite, the fluid acts uniformly on the cylinder walls and does not stick to them. Hence, the fluid pressure is uniform and is independent of the φ and z coordinates. For such loading

$$\theta^0_{r\varphi} = 0, \quad \theta^0_{rz} = 0, \quad \theta^0_{\varphi z} = 0, \quad \theta^0_{zz} = 0 \tag{3.1}$$

Of the first three equations of system (1.2) only one therefore remains, namely,

$$\frac{d\theta^0_{rr}}{dr} + \frac{\theta^0_{rr} - \theta^0_{\varphi\varphi}}{r} = 0 \tag{3.2}$$

To obtain the second equation we will consider the strains

$$\varepsilon^0_r = \frac{du^0_r}{dr}, \quad \varepsilon^0_\varphi = \frac{u^0_r}{r} \tag{3.3}$$

Eliminating u^0_r from both expressions, we obtain the equality

$$\frac{d}{dr}(r\varepsilon^0_\varphi) - \varepsilon^0_r = 0 \tag{3.4}$$

In the linear approximation (in this case this is justified in view of the assumption that the ratio of the cylinder wall thickness to its radius is small), using Hooke's law

$$\varepsilon^0_r = (\sigma^0_r - \nu\sigma^0_\varphi)/E, \quad \varepsilon^0_\varphi = (\sigma^0_\varphi - \nu\sigma^0_r)/E$$

Eq. (3.4) becomes

$$\frac{d}{dr}(r\theta^0_{\varphi\varphi}) - \theta^0_{rr} - \nu\left[\frac{d}{dr}(r\theta^0_{rr}) - \theta^0_{\varphi\varphi}\right] = 0 \tag{3.5}$$

The simultaneous solution of Eqs (3.2) and (3.5) has the form

$$\theta^0_{rr} = D_1 - D_2/\zeta^2, \quad \theta^0_{\varphi\varphi} = D_1 + D_2/\zeta^2 \tag{3.6}$$

where $\zeta = r/R_0$ is the relative radius of the cylinder. The constants D_1 and D_2 are found when satisfying the boundary conditions on the outer and inner surfaces of the cylinder

$$\theta_{rr}|_{\zeta = g} = -p_0, \quad \theta_{rr}|_{\zeta = 1} = -p_1, \quad g = R_1/R_0 \tag{3.7}$$

Substituting (3.6) into conditions (3.7) we obtain

$$D_1 = G\left(p_0 - \frac{p_1}{g^2}\right), \quad D_2 = G(p_0 - p_1)R_0^2, \quad G = \frac{g^2}{1-g^2} \tag{3.9}$$

Hence, the fluid pressure yields an initial stress state in the cylinder wall, described by relations (3.7) and (3.9). It depends both on the geometrical parameters (the cylinder radius and the ratio of the inner and outer radii), and on the nature of the load.

In the special case when the cylinder is acted on solely by an internal pressure ($p_1 = 0$ and $p_0 = p$) the initial stress state in the cylinder walls is as follows:

$$\theta_{rr}^0 = \eta_r p, \quad \eta_r = G\left(1 - \frac{1}{\zeta^2}\right), \quad \theta_{\varphi\varphi}^2 = \eta_\varphi p, \quad \eta_\varphi = G\left(1 + \frac{1}{\zeta^2}\right) \tag{3.10}$$

In Fig. 1 we show graphs of the functions η_r, η_φ against ζ, calculated for different values of the thickness of the cylinder wall. It can be seen that the fluid pressure produces a non-uniform stress state in the cylinder wall, which depends considerably both on the wall thickness and on the cylinder radius.

4. The solution of the boundary-value problem of the oscillations of a uniform cylinder filled with fluid

In the special case when there is no fluid pressure, the solution of boundary-value problem (1.2) and (1.3), after using an inverse Fourier transformation, can be represented in the form ($\mathbf{u} = (u_r, u_z)$ is the displacement vector and $\mathbf{q} = (q_r, q_z)$ is the stress vector)

$$\mathbf{u}(r,z) = \frac{1}{2\pi}\int_{-a}^{a} \mathbf{k}(z-\xi, r, \omega)\mathbf{q}(\xi)d\xi, \quad \mathbf{k}(s, r, \omega) = \int_\Gamma \mathbf{K}(\alpha, r, \omega)e^{i\alpha s}d\alpha \tag{4.1}$$

The elements of the matrix function $\mathbf{K}(\alpha,r,\omega)$ are defined by the formulae ($k = 1, 2$)

$$K_{1k} = \Delta_0^{-1}\sum_{n=1}^{2}[\Delta_{k,2n-1}I_1(\sigma_n r) + \Delta_{k,2n}K_1(\sigma_n r)]$$

$$K_{2k} = -i\alpha\Delta_0^{-1}\sum_{n=1}^{2}\sigma_n^{-1}[\Delta_{k,2n-1}I_0(\sigma_n r) + \Delta_{k,2n}K_0(\sigma_n r)]$$

$$K_{3k} = i\Delta_0^{-1}\Delta_{k5}I_0(\gamma r) \tag{4.2}$$

Here Δ_0 is a determinant and Δ_{ik} is the cofactor of an element of the matrix

$$\mathbf{L} = \begin{Vmatrix} l_{11}^0 & l_{12}^0 & l_{13}^0 & l_{14}^0 & 0 \\ l_{21}^0 & l_{22}^0 & l_{23}^0 & l_{24}^0 & 0 \\ l_{11}^1 & l_{12}^1 & l_{13}^1 & l_{14}^1 & l_{35}^1 \\ l_{21}^1 & l_{22}^1 & l_{23}^1 & l_{24}^1 & 0 \\ l_{51}^1 & l_{52}^1 & l_{53}^1 & l_{54}^1 & l_{55}^1 \end{Vmatrix} \tag{4.3}$$

$$l_{sk}^0 = l_{sk}(R_0),\ s = 1, 2,\ k = 1, 2, 3, 4,\ l_{sk}^1 = l_{sk}(R_1),\ s = 1, 2, 5,\ k = 1, 2, 3, 4, 5$$

$$l_{11}(r) = \theta\sigma_1^{-1}I_0(\sigma_1 r) - r^{-1}I_1(\sigma_1 r), \quad l_{12}(r) = -\theta\sigma_1^{-1}K_0(\sigma_1 r) - r^{-1}K_1(\sigma_1 r)$$

$$l_{13}(r) = \sigma_2 I_0(\sigma_2 r) - r^{-1}I_1(\sigma_2 r), \quad l_{14}(r) = -\sigma_2 K_0(\sigma_2 r) - r^{-1}K_1(\sigma_2 r)$$

$$l_{21}(r) = \alpha^2 I_1(\sigma_1 r), \quad l_{22}(r) = \alpha^2 K_1(\sigma_1 r), \ l_{23}(r) = \theta I_1(\sigma_2 r), \quad l_{24}(r) = \theta K_1(\sigma_2 r)$$

$$l_{35} = \kappa_2\rho' \upsilon I_0(\gamma r)/2, \quad l_{51}(r) = I_1(\sigma_1 r), \quad l_{52}(r) = K_1(\sigma_1 r)$$

$$l_{53}(r) = I_1(\sigma_2 r), \quad l_{54}(r) = K_1(\sigma_2 r), \quad l_{55}(r) = -\gamma\lambda_1 I_1(\gamma r)/(2\pi)$$

$\lambda_1 = 2\pi C\omega^{-1}$ is the wavelength of a sound wave in the fluid and $\theta = \alpha^2 - \kappa_2^2/2$.

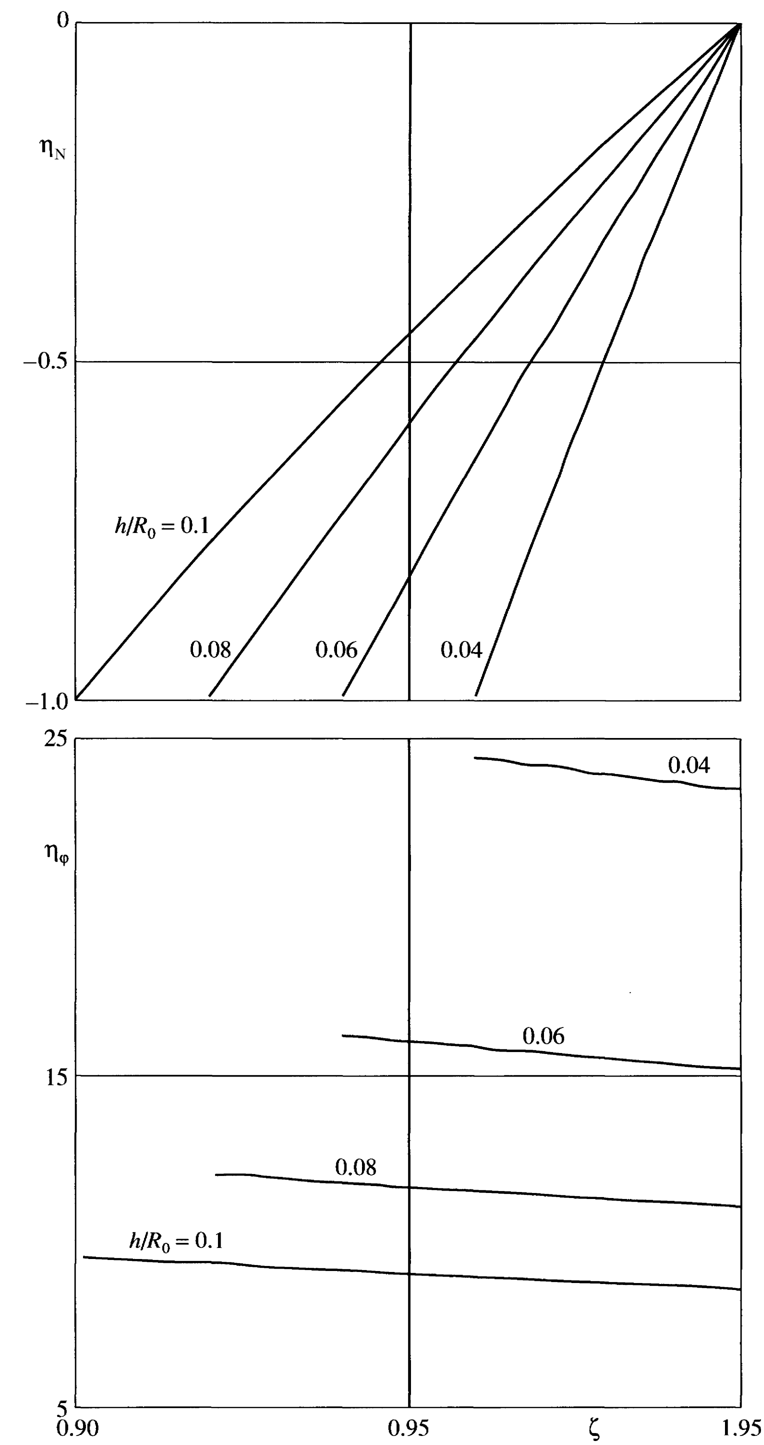

Fig. 1.

5. The solution of the problem of the oscillations of a cylinder filled with a fluid at a high pressure

When the stress state is non-uniform, the solution of boundary-value problem (1.2), (1.3) can be represented in the form (4.1) (Refs 1, 2), but the kernel symbol is constructed numerically. The elements of the matrix function $\mathbf{K}(\alpha, r, \kappa_2)$ (4.2) have the following structure:

$$K_{mn} = \frac{1}{\Delta_0}\sum_{i=1}^{4}\Delta_{in}y_{m+2,i}, \quad m, n = 1, 2 \tag{5.1}$$

Here Δ_0 is a determinant and Δ_{in} is the cofactor of element l_{in} of the 5 × 5 matrix $\mathbf{L} = ||l_{ij}||$. The elements of the matrix are defined by the formulae ($m = 1, 2, 3, 4$)

$$l_{15} = l_{25} = l_{45} = l_{51} = l_{52} = l_{54} = 0$$

$$l_{m1} = l_m^+(R_0), \quad l_{m2} = l_m^-(R_0), \quad l_{m3} = l_m^+(R_1), \quad l_{m4} = l_m^-(R_1)$$

$$l_m^+(r) = \lambda_m y_{m1}(\alpha, r) + \lambda r^{-1} y_{m3}(\alpha, r) - \lambda y_{m4}(\alpha, r)$$

$$l_m^-(r) = \mu^+ y_{m2}(\alpha, r) + \alpha^2 \mu^- y_{m3}(\alpha, r)$$

$$l_{35} = \rho' \kappa_2 \upsilon I_0(\gamma R_0), \quad l_{53} = -1, \quad l_{55} = (2\pi)^{-1} \gamma \lambda_1 I_1(\gamma R_0) \tag{5.2}$$

The functions $y_{ij}(\alpha, r)$ ($i,j = 1, 2, 3, 4$) are the set of linearly independent solutions of the system of equations

$$\mathbf{Y}' = \mathbf{M}(\alpha, r)\mathbf{Y} \tag{5.3}$$

with initial conditions $y_{ij}(\alpha, R_0) = \delta_{ij}$ (δ_{ij} is the Kronecker delta) and the 4 × 4 matrix

$$\mathbf{M} = \lambda_m^{-1} \|m_{ij}\|_{i,\, j = 1}^4 \tag{5.4}$$

with elements

$$m_{11} = -\frac{1}{r}, \quad m_{12} = \frac{\lambda^+}{\lambda_m}, \quad m_{13} = \frac{\sigma_2^2 \mu^-}{\lambda_m} + \frac{1}{r^2}, \quad m_{14} = 0$$

$$m_{21} = -\frac{\alpha^2 \lambda^-}{\mu^-}, \quad m_{22} = -\frac{(\mu^-)'}{\mu^-} + \frac{1}{r}, \quad m_{23} = -\frac{\alpha^2((\mu^-)' + \lambda^-)}{\mu^-}, \quad m_{24} = \sigma_1^2$$

$$m_{31} = m_{42} = 1, \quad m_{3n} = 0, \quad n = 2, 3, 4, \quad m_{4n} = 0, \quad n = 1, 3, 4 \tag{5.5}$$

6. The integral equation of the problem of the radial oscillations of a bandage on the cylinder

The integral representation (4.1) with the function (4.2) or the function (5.1) describes the displacement of an arbitrary point of the cylinder, made of a material of uniform thickness (when there are no initial stresses), or non-uniform thickness (when there are initial stresses), respectively. In the case of the problem of radial oscillations of a bandage on the cylinder, it is necessary to put $r = R_0$ in Eqs (4.2) and (5.1), the displacement of the inner surface of the bandage (the surface of the cylinder in the contact area) is assumed to be known. We will further assume that there is no friction in the contact area. Equality (4.1) in this case degenerates into a scalar integral equation in the unknown distribution function of the contact stresses under the bandage

$$u_r(R_0, z) = \frac{1}{2\pi} \int_{-a}^{a} k_{11}(z - \xi, R_0, \kappa_2) q_r(\xi) d\xi, \quad k_{11}(s, R_0, \kappa_2) = \int_\Gamma K_{11}(\alpha, R_0, \kappa_2) e^{i\alpha s} d\alpha \tag{6.1}$$

In the case of a uniform material

$$K_{11}(\alpha, R_0, \kappa_2) = \Delta_0^{-1} \sum_{k=1}^{2} [\Delta_{1,\,2k-1} I_1(\sigma_k R_0) + \Delta_{1,\,2k} K_1(\sigma_1 R_0)] \tag{6.2}$$

The functions Δ_0 and Δ_{ik} are defined by formulae (4.2) and (4.3) (the presence of the fluid is taken into account) with the condition that $r = R_0$. In the case of a non-uniform medium the function K_{11} is constructed numerically using formulae (5.1)–(5.4).

The contour Γ in representation (6.1) is chosen from the rules given previously in Refs 11 and 12. It coincides almost everywhere with the real axis and deviates from it when going round positive poles downwards and negative poles upwards. A feature of the class of problems considered is the presence in the function K_{11} (6.2) of real zeros and poles, which cause an oscillation of the kernel of the integral equation.[10–14] The following values of the parameters are used: the cylinder is made of steel ($\rho = 7.748 \times 10^3$ kg/m^3, $\lambda = 1.1 \times 10^{11}$ N/m^2 and $\mu = 0.804 \times 10^{11}$ N/m^2) and the fluid is petroleum ($\rho_1 = 0.8 \times 10^3$ kg/m^3 and $c_1 = 1400$ m/s).

In Fig. 2 we show dispersion curves for the empty cylinder (the first two "elastic" modes – the continuous curves) and for a cylinder filled with the fluid (the first five modes – the dashed curves, indicated by the numbers 0–4). As follows from the graphs the fluid considerably changes the dispersion properties of the cylinder: the number of propagating wave modes is increased considerably and a zeroth low-velocity mode possessing weak dispersion appears. All subsequent modes have a complex dispersion form. The "elastic" ranges are frequency ranges in which the dispersion curves of the filled cylinder are identical with the "elastic" branches – the dispersion curves of the empty cylinder, and alternate with "fluid" ranges – frequency ranges in which the curves of the filled cylinder, on changing from one elastic branch to another, coincide with the "fluid" branches – the dispersion curves of the "liquid cylinder" – an acoustic cylinder with the parameters (density and velocity of sound) of the fluid. At the instants of transition from the elastic branch to the fluid branch considerable dispersion occurs with a sharp reduction in velocity. After separating from the last elastic branch, the mode of the curve of the filled cylinder becomes a purely fluid mode with weak dispersion. The dispersion diagram of the filled cylinder as a whole is a combination of dispersion diagrams of the empty and fluid cylinders.

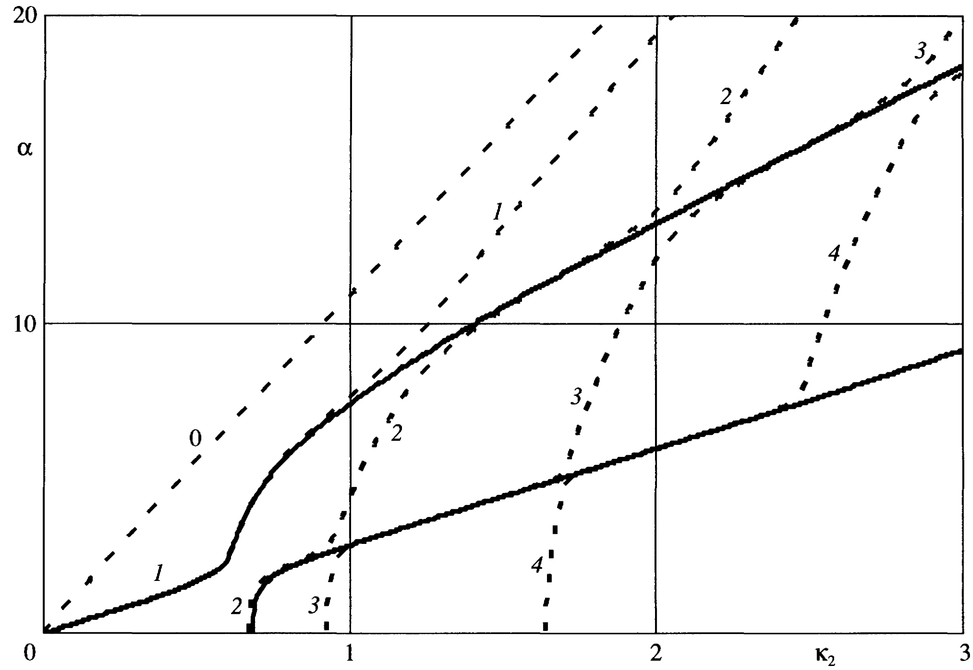

Fig. 2.

In Fig. 3 we show curves of $\eta(\kappa_2)(\eta=\alpha^0-\alpha^\sigma;\ \alpha^0)$ and α^σ are the values of the wave numbers when there is no prestress and when there is prestress respectively), which illustrates the effect on the initial stresses on the first modes (the first mode is in the upper part of Fig. 3 and the second mode is in the lower part) of the surface waves of the empty cylinder (the continuous curves) and the cylinder filled with fluid (the dashed curves). The numbers 1 and 2 indicate curves calculated for initial extensions of 0.01 and 0.05 respectively. As follows from the graphs the effect of the initial stresses has a complex form, depending both on the presence of the fluid and on the mode. For an empty cylinder there are frequencies at which the effect of the initial stresses is either a maximum (e_1 is the first mode and e_2 is the second mode respectively), or zero (e_3 on the first mode). For the first mode in the range $[0,e_3]$ the presence of initial stresses leads to a reduction in the wave numbers, and above e_3 it leads to an increase in their values. If we take into account the fact that the relative phase velocity is given by the expression $V=\kappa_2\alpha^{-1}$, then in the range $[0,e_3]$ it is increased and above e_3 it is reduced. The situation is different with the second mode, for which the action of the initial stresses over the whole range leads to an increase in the wave numbers, and thereby to a reduction in the phase velocity.

The presence of the fluid considerably changes the nature of the effect of the initial stresses on the dispersion properties of the elastic cylinder. In particular it shows itself to the greatest extent only in limited frequency ranges of "elasticity" (the range $[0,f_1]$ for the first mode and the range $[f_2,f_3]$ for the second mode). It has no effect in the "fluid" ranges. For a filled cylinder there are also frequencies at which the effect of the initial stresses is either a maximum (e_1 for the first mode and e_2 for the second mode), or it has no effect (f_1 for the first and f_3 and f_4 for the second mode). Hence, the values of the frequencies at which the effect of the initial stresses is a maximum do not depend on the presence of the fluid.

It follows from the graphs, as a whole, that the initial stresses only have an effect on the elastic component of the wave process. A maximum is reached either at points of inflection of the dispersion curve (the first mode), or in the neighbourhood of the critical frequency at which an elastic mode occurs.

An analysis of the distribution of the poles and zeros of the kernel symbol of the integral operator of the problem of partial oscillations of the bandage on the cylinder filled with fluid showed that there is a strict alternation of the zeros and poles which ensures uniqueness of the solution of the integral equation.[14,15] Here the presence of a considerable number of zeros and poles indicates considerable oscillations of the kernel, which determines the need, when solving integral Eq. (6.1), to use methods which enable this fact to be taken into account.[13–17]

7. Solution of the integral equation

The Fourier transform of the solution of integral Eq. (6.1) is represented by the formula[17]

$$Q(\alpha) = \frac{T(\alpha)}{\Pi(\alpha)} + \sum_{k=1}^{2M} C_k e^{i\alpha z^p}, \quad \Pi(\alpha) = \prod_{k=1}^{M} \frac{(\alpha^2-\gamma_k^2)}{(\alpha^2-\xi_k^2)} \tag{7.1}$$

where ξ_k ($k=1, 2, \ldots, n_1$) and γ_k ($k=1, 2, \ldots, n_2$) are real poles and zeros of the symbol K_{11} (α, R_0, κ_2), and the remaining ξ_k ($k=n_1+1, \ldots, M$) and γ_k ($k=n_2+1, \ldots, M, M\geq\max\{n_1, n_2\}$) are complex poles and zeros of K_{11} (α, R_0, κ_2), lying in the range $|\mathrm{Im}\ \alpha|\leq\varepsilon_0$.

The function $T(\alpha)$ is the Fourier transform of the function

$$t(z) = t_0(z) + \sum_{k=1}^{2M} C_k t_k(z), \quad t_k(z) = \sum_{p=1}^{N} \beta_k^p \psi_p(z), \quad k = 0, 1, \ldots, 2M \tag{7.2}$$

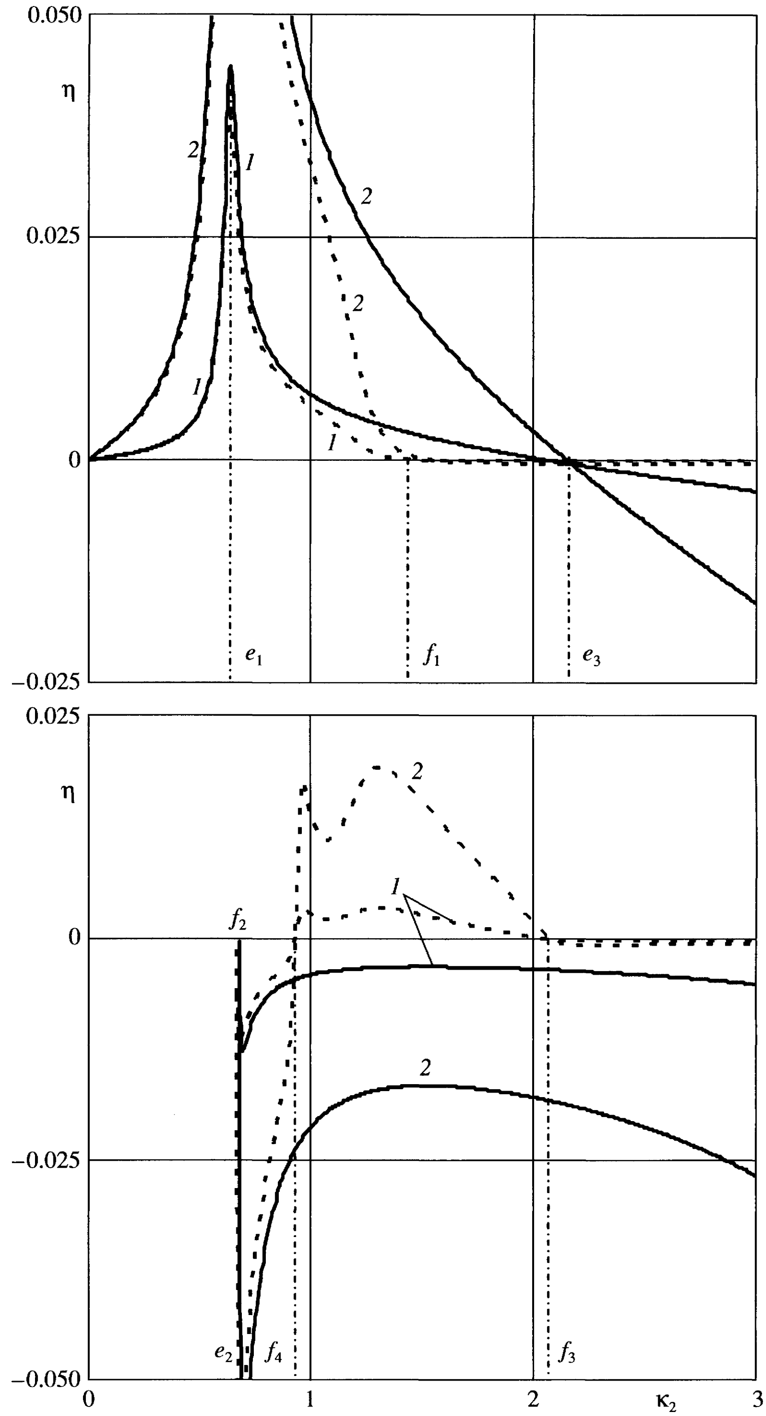

Fig. 3.

Here $\psi_p(z)$ is a system of coordinate functions, specified in the interval $[-a, a]$, and the coefficients β_k^p satisfy the system of $2M+1$ algebraic equations

$$\mathbf{A}\mathbf{B}_k = \mathbf{F}_k, \quad k = 0, 1, \ldots, 2M$$

$$\mathbf{A} = \|A_{pl}\|_{p,l=1}^{N}, \quad A_{pl} = \int_{\Gamma} K_0(\alpha)\Psi_p(\alpha)\Phi_l^*(\alpha)d\alpha, \quad K_0(\alpha) = \frac{K(\alpha)}{\Pi(\alpha)}$$

$$\mathbf{F}_k = \{f_k^l\}_{l=1}^{N}, \quad f_0^l = \int_{-a}^{a} f(z)\varphi_l(z)dz, \quad f_k^l = \int_{\Gamma} K(\alpha)\Phi_l(\alpha)e^{-i\alpha z^k}d\alpha, \quad \mathbf{B}_k = \{\beta_k^p\}_{p=1}^{N} \tag{7.3}$$

Here $\Psi_p(\alpha)$ and $\Phi_l(\alpha)$ are Fourier transforms of the systems of coordinate functions $\psi_p(z)$ and $\varphi_l(z)$. The asterisk denotes a complex conjugate quantity. The constants C_k ($k = 1, 2, \ldots, 2M$) satisfy the algebraic system

$$\sum_{p=1}^{N} \beta_0^p \Psi_p(\pm\gamma_n) - \sum_{k=1}^{2M} C_k \sum_{p=1}^{N} \beta_k^p \Psi_p(\pm\gamma_n) = 0, \quad n = 0, 1, \ldots, 2M \tag{7.4}$$

8. The effect of the prestressed state of the cylinder on the stress distribution in the contact area

Formulae (7.1)–(7.4) enable us to investigate the effect of the cylinder parameters on the wave process in the contact area. The effect of the presence of fluid and the initial stresses in the cylinder wall on the distribution of the contact stresses (the real component) is illustrated by the graphs of Re$q(z)$ in the upper part of Figs. 4 and 5. The continuous curves correspond to an empty cylinder and the dashed curves

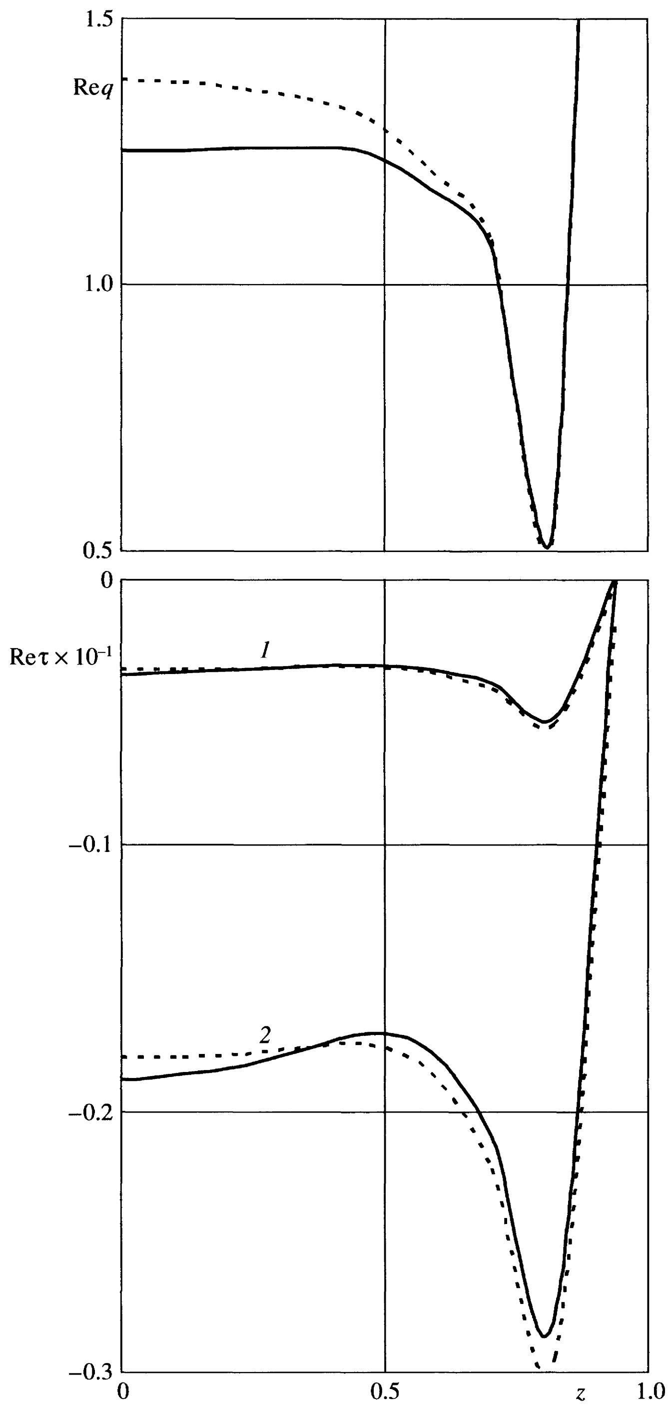

Fig. 4.

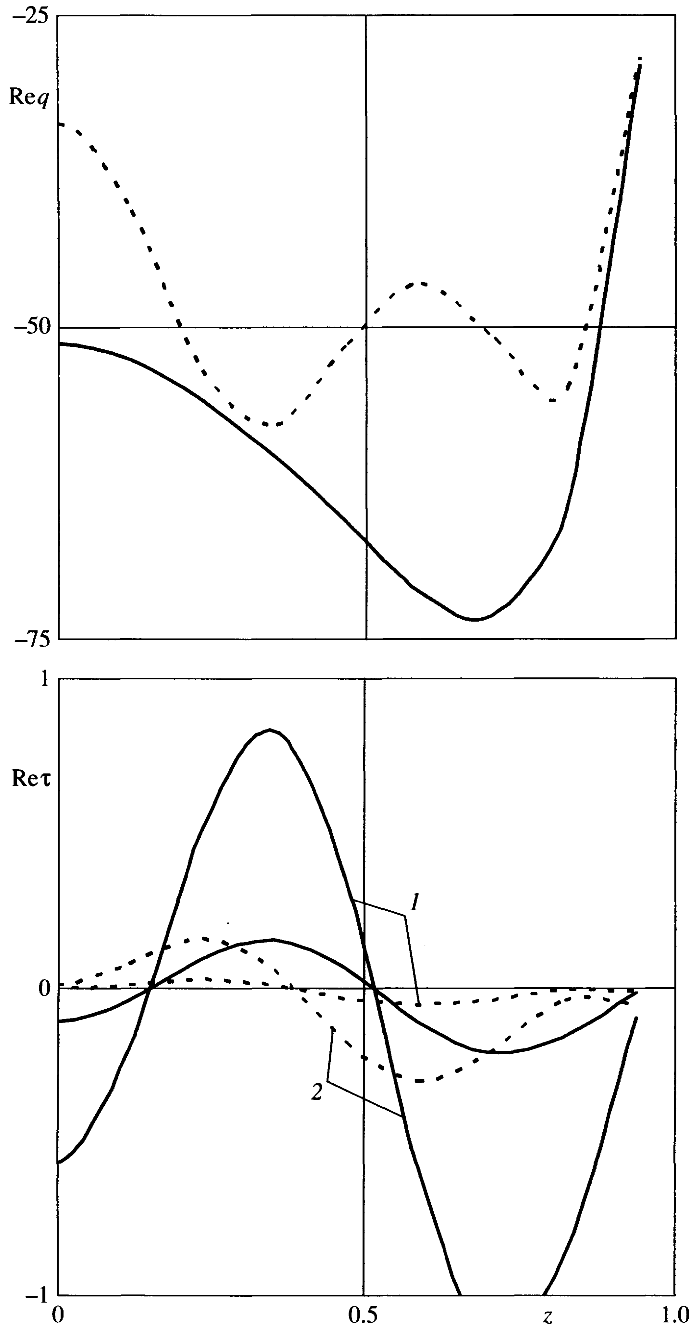

Fig. 5.

correspond to a cylinder filled with fluid. As follows from the graphs, the presence of fluid only affects the distribution of the contact stresses at high frequencies. At low frequencies the effect of the fluid is imperceptible.

The effect of the initial stresses on the formation of the field of contact stresses (the real component $\tau = q_0 - q_\sigma$, where q_0 and q_σ are the contact stresses in the natural and initially-deformed state) is illustrated by the graphs in the lower part of Figs. 4 and 5. The numbers 1 and 2 indicate the curves for an initial extension $\nu_1 = 0.01$ and $\nu_1 = 0.05$ respectively. As follows from the graphs, the effect of the initial stresses increases considerably as the oscillation frequency increases. It can reach several percent at high frequencies. As follows from the graphs, at low frequencies the role of the fluid in forming the wave field in the contact area is insignificant. At the same time, at high frequencies the fluid participates to a considerable extent in the formation of the wave field in the prestressed cylinder. In this case the role of the fluid manifests itself in a certain weakening of the oscillating form of the effect of the initial stresses. Hence, we have shown that it is necessary to take the presence of the fluid into account when monitoring the initial stresses at high frequencies. At low frequencies the part played by the fluid in forming the wave field is insignificant. This indicates the need, in each specific case, to make a detailed investigation of the dynamic properties of the object, taking into account the frequency spectrum of the possible action and presence of the fluid.

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